International Journal of Theoretical Physics, Vol. 2, No. 2 (1969), pp. 101–123.

What is a Hidden Variable Theory of Quantum Phenomena?

JEFFREY BUB†

University of Minnesota

Abstract

The purpose of this paper is to clarify the relationship between existing so-called 'hidden variable theories' of quantum phenomena and some well-known proofs, such as those of von Neumann, Jauch and Piron, and Kochen and Specker, which purport to establish that no such theory is possible. The proof of Kochen and Specker, which is a stronger version of von Neumann's result, demonstrates the impossibility of embedding the algebraic structure of physical parameters of the quantum theory, represented by the self-adjoint Hilbert space operators, into the commutative algebra of real-valued functions on a 'phase space' of hidden states. This is a necessary condition for a hidden variable extension of the quantum theory in the usual sense of a statistical mechanical derivation of the statistical theorems of the quantum theory in the classical manner. No existing so-called 'hidden variable theory' is a counter-example to von Neumann's proof. The early 1951 'hidden variable theory' of Bohm and the recent theory of Bohm and Bub are not in fact hidden variable theories in the usual sense of the term. Since the term 'hidden variable theory' is justifiably used to denote the kind of theory rejected by von Neumann, Jauch and Piron, and Kochen and Specker, it is suggested that the term should not be used as a label for the theories considered by Bohm and other workers in this field. Such theories could be regarded as fundamentally compatible with the original Copenhagen interpretation of the quantum theory, as expressed by Bohr.

1. *Introduction*

What is a hidden variable theory of quantum phenomena? On the one hand there are well-known proofs, such as those of von Neumann (1955), Jauch & Piron (1963) and Kochen & Specker (1967a) which purport to establish that no such theory is possible. On the other hand, there exist so-called 'hidden variable theories' (Bohm & Bub, 1966) which, at least at first sight, seem to provide counter-examples to these theorems. In spite of the excellent analysis by Bell (1966), the relationship between these proofs and the 'hidden variable theories' is still obscure.‡ The purpose of this paper will be to clarify this relationship. The question is further confused by current misconceptions about the role of measurement in the quantum theory, partly resulting from a misunderstanding of Bohr's principle of complementarity, the core of the original Copenhagen interpretation.

The proof of Kochen and Specker demonstrates the impossibility of embedding the algebraic structure of physical parameters of the quantum theory, represented by the self-adjoint Hilbert space operators, into the

† Supported by the National Science Foundation.

‡ See, for example, Letters to the Editor, *Reviews of Modern Physics*, **40**, 228 (1968).

commutative algebra of real-valued functions on a 'phase space' of hidden states. This turns out to be a necessary condition for a hidden variable extension of the quantum theory, as understood by Kochen and Specker. von Neumann's original proof is a much weaker result which excludes only a relatively uninteresting class of hidden variable theories. In its present form, the Jauch and Piron proof, in terms of the lattice of quantum theoretical propositions, is not an interesting improvement on von Neumann's result, but the Kochen and Specker proof suggests that a lattice theoretical reconstruction in a stronger form is possible.

No existing so-called 'hidden variable theory' is a counter-example to the stronger version of von Neumann's proof. The early 1951 'hidden variable theory' of Bohm (1952a), and the recent theory of Bohm & Bub (1966), are not in fact hidden variable theories in the usual sense of the term. Since the term 'hidden variable theory' is justifiably used to denote the kind of theory rejected by von Neumann, Jauch and Piron, and Kochen and Specker, it would perhaps be appropriate to drop the use of the term as a label for the theories considered by Bohm and other workers in this field.† Such theories could be regarded as fundamentally compatible with the original Copenhagen interpretation of the quantum theory, *as expressed by Bohr.*

The conclusions of this paper will therefore be that there are no hidden variable theories of quantum phenomena in the usual sense, that the term 'hidden variable theory' for the kind of theory considered by Bohm and his collaborators is unfortunate and misleading, and that this latter approach might well be characterized as an extension of Bohr's conception of wholeness as opposed to the von Neumann philosophy.

2. *The Proof of Kochen and Specker*

The quantum theory, as a statistical theory, may be characterized by a set $\mathscr{S}$ of statistical states and a set $\mathscr{P}$ of physical parameters. Each $\psi \in \mathscr{S}$, $A \in \mathscr{P}$ defines a probability measure $\mu_{\psi A}$ on the real line $\mathscr{R}$, i.e., if $\mathscr{M}$ is a measurable subset of $\mathscr{R}$, then $\mu_{\psi A}(\mathscr{M})$ is the probability that the value of the physical parameter A in the statistical state ψ lies in the set $\mathscr{M}$. It follows that the average value of A in the state ψ is given by the expression:

$$\mathrm{Av}_{\psi}(A) = \int_{-\infty}^{\infty} r \, d\mu_{\psi A}(r) \tag{2.1}$$

If the physical parameters are interpreted as denoting physical attributes of objects in the usual sense, then the question arises for any such theory: Is it possible to embed the statistical theory into a more fundamental

† Bohm has suggested the term 'contingent parameters' instead of 'hidden variables' in an unpublished paper presented to a Symposium on the foundations of quantum theory ('Quantum Theory and Beyond'), held at the University of Cambridge, England, July 15–20, 1968.

theory, so that the probability measures $\mu_{\psi A}$ (for a fixed $\psi \in \mathscr{S}$, and all $A \in \mathscr{P}$) can be associated with a probability distribution over certain 'hidden' states, which are not themselves statistically related to the physical parameters. This embedding is understood in the following sense:†

(I). Each statistical state $\psi \in \mathscr{S}$ is associated with a probability measure ρ_ψ on the 'phase space' $\mathscr{X}$ of hidden states, i.e., if Λ is a measurable subset of $\mathscr{X}$, then $\rho_\psi(\Lambda)$ is the probability that the state of the system (the phase point of the system) lies in Λ.

(II). Each physical parameter $A \in \mathscr{P}$ is associated with a function $f_A: \mathscr{X} \to \mathscr{R}$, mapping the set $\mathscr{X}$ of hidden states into the reals. In accordance with the interpretation of the physical parameters as denoting physical attributes of objects, it is assumed that the value of the physical parameter $g(A)$ is equal to $g(a)$ if a is the value of the parameter A, i.e., that $f_{g(A)}(\lambda) = g(a)$ if $f_A(\lambda) = a$, or:

$$f_{g(A)} = g(f_A) \tag{2.2}$$

(III). The measure of the set of phase points in $\mathscr{X}$ which are mapped by the function f_A onto the set $\mathscr{M}$ of reals is equal to the appropriate measure of the set $\mathscr{M}$ specified by the original statistical theory, i.e.:

$$\mu_{\psi A}(\mathscr{M}) = \rho_\psi(f_A^{-1}(\mathscr{M})) \tag{2.3}$$

Equivalently:

$$\mathrm{Av}_\psi(A) = \int_{\mathscr{X}} f_A(\lambda)\, d\rho_\psi(\lambda) \tag{2.4}$$

Evidently, this embedding of the algebraic structure of physical parameters of the original statistical theory into the algebra of real-valued functions on a suitable phase space, in such a way that the statistical conditions (2.3) and (2.4) are satisfied, may or may not be possible. This is a purely mathematical question, the answer to which depends on the specific algebraic structure of the statistical theory involved. Two points should be emphasized here. Firstly, the possibility of the embedding is a necessary, but not sufficient, condition for the construction of a deterministic theory which will include the quantum theory. Even if a suitable embedding exists, it is a further problem whether or not a deterministic equation of motion can be found for the hidden states which will be consistent with the time transformations of the quantum theory. Secondly, the possibility of the embedding is not even a necessary condition for the construction of a deterministic theory which will correctly explain *all the experimental results currently explained by the quantum theory*. For example, such a theory need not necessarily satisfy (III), i.e., we could have: $\rho_\psi(f_A^{-1}(\mathscr{M})) = \mu'_{\psi A}(\mathscr{M})$, where $\mu'_{\psi A}$ is experimentally indistinguishable from $\mu_{\psi A}$ for those experiments which have actually been performed.

Kochen and Specker demonstrate that the embedding is impossible for the quantum theory. The proof is carried out in the following way: A set of

† The following conditions have been abstracted from the article by Kochen & Specker (1967a).

physical parameters A_i is called *commeasurable* if there exists a physical parameter B and functions α_i, such that $A_i = \alpha_i(B)$. It is a theorem that parameters of the quantum theory represented by commuting Hilbert space operators are commeasurable. A set of physical parameters is said to form a *partial algebra* if sums and products are defined for *commeasurable* parameters according to the rules:

$$\begin{aligned} k_1 A_1 + k_2 A_2 &= (k_1 \alpha_1 + k_2 \alpha_2)(B) \\ A_1 A_2 &= (\alpha_1 \alpha_2)(B) \end{aligned} \tag{2.5}$$

where $A_1 = \alpha_1(B)$, $A_2 = \alpha_2(B)$, and k_1, k_2 are real numbers. The partial operations are preserved under a map f which satisfies the condition (2.2), i.e.

$$\begin{aligned} f_{k_1 A_1 + k_2 A_2} &= f_{(k_1 \alpha_1 + k_2 \alpha_2)(B)} \\ &= (k_1 \alpha_1 + k_2 \alpha_2)(f_B) \\ &= k_1 \alpha_1(f_B) + k_2 \alpha_2(f_B) \\ &= k_1 f_{\alpha_1(B)} + k_2 f_{\alpha_2(B)} \\ &= k_1 f_{A_1} + k_2 f_{A_2} \\ f_{A_1 A_2} &= f_{(\alpha_1 \alpha_2)(B)} \\ &= (\alpha_1 \alpha_2)(f_B) \\ &= \alpha_1(f_B)\, \alpha_2(f_B) \\ &= f_{\alpha_1(B)} f_{\alpha_2(B)} \\ &= f_{A_1} f_{A_2} \end{aligned} \tag{2.6}$$

The set $\mathscr{R}^{\mathscr{X}}$ of all functions $f: \mathscr{X} \to \mathscr{R}$ from the phase space of hidden states into the reals forms a commutative algebra over $\mathscr{R}$. Since condition (2.2) implies that the partial operations are preserved under the map f, a necessary criterion for a hidden variable extension of the quantum theory [satisfying conditions (I), (II), and (III)] is the existence of an embedding of the partial algebra $\mathscr{Q}$ of self-adjoint Hilbert space operators (representing the physical parameters of the theory) into the commutative algebra $\mathscr{R}^{\mathscr{X}}$. Each hidden state $\lambda \in \mathscr{X}$ may be regarded as defining a homomorphism $h_\lambda: \mathscr{Q} \to \mathscr{R}$ of the partial algebra $\mathscr{Q}$ into $\mathscr{R}$, namely the homomorphism defined by $h_\lambda(A) = f_A(\lambda)$. Thus, the possibility of the embedding implies the existence of functions h_λ, which simultaneously associate a value with every physical parameter.

Kochen and Specker prove that there does not exist such a homomorphism $h: \mathscr{Q} \to \mathscr{R}$. The proof is carried out by showing that there is no homomorphism from the partial Boolean algebra $\mathscr{B}$ of quantum theoretical propositions† represented by the idempotent elements of $\mathscr{Q}$ onto $\{0, 1\}$.

† See Section 4, below, for an explication of the use of the term 'proposition' in this sense.

This is equivalent to demonstrating that the partial Boolean algebra of quantum theoretical propositions cannot be embedded into a Boolean algebra. Specifically, Kochen and Specker demonstrate that there does not exist a homomorphism $h:\mathscr{D} \to \{0,1\}$ from a certain finite partial Boolean sub-algebra $\mathscr{D}$ of $\mathscr{B}$ onto $\{0,1\}$. The proof is not trivial because some partial algebras can be embedded into commutative algebras. It is a consequence of its peculiar algebraic structure (i.e., the algebraic structure of self-adjoint operators in Hilbert space) that the partial algebra of physical parameters of the quantum theory cannot be embedded into the commutative algebra $\mathscr{R}^{\mathscr{X}}$.

The aim of Kochen and Specker was to prove the non-existence of a hidden variable extension of the quantum theory, in a certain sense. In order to appreciate the significance of their result, it is necessary to take a closer look at the definition of 'commeasurability' of physical parameters, coupled with condition (2.2). Kochen and Specker refer to the physical parameters of the quantum theory as 'observables'. The term 'commeasurable' is introduced to suggest 'simultaneously measurable': 'commeasurable observables' $A_i = \alpha_i(A)$ are 'simultaneously measurable', because the values of the 'observables' A_i are obtained by applying the functions α_i to the measured value of the 'observable' A. This suggests that 'commeasurable observables' are interpreted as 'co-determined attributes of the object measured'. Condition (2.2), $f_{g(A)} = g(f_A)$, is therefore a necessary condition for any reasonable hidden variable extension of the quantum theory, since it simply expresses the natural requirement that functionally related or co-determined attributes of an object should be associated with correspondingly related real-valued functions on the phase space in the hidden variable extension of the statistical theory.

However, Kochen and Specker identify the physical parameters of the quantum theory with the self-adjoint Hilbert space operators. Now, it is possible to have a set of operators $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$, such that $\mathbf{AB} = \mathbf{BA}$, $\mathbf{BC} = \mathbf{CB}$, but $\mathbf{AC} \neq \mathbf{CA}$. This means that the set (A,B) of corresponding physical parameters is 'commeasurable', the set $\{B,C\}$ is 'commeasurable', but the set $\{A,C\}$ is not 'commeasurable'. Thus, there is a physical parameter A' such that A and B are expressible as functions of A', and a physical parameter C' such that B and C are expressible as functions of C', but there does not exist a physical parameter P such that A and C are expressible as functions of P. If $B = g_1(A') = g_2(C')$, then the 'commeasurable' pair $\{B,A'\}$ cannot be interpreted as a 'co-determined pair of attributes of an object', since the pair $\{B,C'\}$ would then have to be similarly interpreted, implying that the pair $\{A',C'\}$ is co-determined. But A' and C' are 'non-commeasurable' and correspond to non-commuting operators.

For example, suppose A corresponds to the operator $\mathbf{A} = \sum_i a_i \mathbf{P}_i$ and C corresponds to the operator $\mathbf{C} = \sum_j c_j \mathbf{Q}_j$, where $\mathbf{P}_i$, $\mathbf{Q}_j$ are projection operators onto complete, orthogonal sets of vectors, and a_i, c_j are the eigenvalues of $\mathbf{A}$ and $\mathbf{C}$ respectively. (It is assumed that the Hilbert space has more than two dimensions.) Let $\mathbf{P}_1 = \mathbf{Q}_1$ and define $\mathbf{B} = \mathbf{P}_1 = \mathbf{Q}_1$. Then the idempotent physical parameter B can be expressed as a function of A, i.e., $B=g_1(A)$, defined by $g_1(a_1)=1$ and $g_1(a_i)=0$ $(i\neq 1)$. Similarly, B can be expressed as a function of C, i.e., $B=g_2(C)$, defined by $g_2(c_1)=1$ and $g_2(c_j)=0$ $(j\neq 1)$. If $B=1$ is associated with the proposition asserting that the 'quantum object' possesses the attribute symbolized by $A=a_1$, then $B=1$ must with equal validity be associated with the proposition asserting that the 'quantum object' possesses the attribute symbolized by $C=c_1$. But this contradicts the fact that A and C do not commute, i.e., that A and C are 'non-commeasurable'.

Since the self-adjoint Hilbert space operators representing the physical parameters of the quantum theory cannot be interpreted as referring to the 'attributes of quantum objects', the term 'commeasurable' to describe a functional relationship between certain parameters is misleading. A more neutral term would be 'coherent'. This term does not prejudice the interpretation of the physical parameters of the theory and will be used below instead of the term 'commeasurable'. Condition (2.2) now becomes suspect. Why should coherent physical parameters be associated with related real-valued function on a phase space of hidden states? For a particular $\lambda\in\mathscr{X}$ such that $f_B(\lambda)=1$, this would imply that $f_A(\lambda)=a_1$ and $f_C(\lambda)=c_1$. This is not a theorem in the quantum theory. It is quite possible, for example, that $f_A(\lambda)=a_1$ and $f_C(\lambda)\neq c_1$ for some $\lambda\in\mathscr{X}$, and also that $\mu_{\psi A}(\{a_1\})=\rho_\psi(f_A^{-1}(a_1))=\mu_{\psi B}(\{1\})=\rho_\psi(f_B^{-1}(1))=\mu_{\psi C}(\{c_1\})=\rho_\psi(f_C^{-1}(c_1))$, i.e., that the statistical relations are satisfied. In other words, condition (2.2) implies that $f_B=g_1(f_A)$ and $f_B=g_2(f_C)$, which imposes the restriction $g_1(f_A)=g_2(f_C)$ on the phase space functions associated with the non-commuting operators **A**, **C**.

This objection to condition (2.2) does not yet go to the heart of the matter. If the physical parameters of the quantum theory, represented by self-adjoint Hilbert space operators, are not interpreted as referring to the attributes or properties of 'quantum objects', then there is no motivation whatsoever for developing a hidden variable extension of the quantum theory *in the sense defined by Kochen and Specker*. The hidden variable problem as posed by von Neumann and refined by Kochen and Specker is a pseudo-problem. As Kochen and Specker themselves point out (Kochen & Specker, 1967b), the hidden variable problem without condition (2.2) is trivial: it is always possible mathematically to introduce a phase space $\mathscr{X}$, and to associate real-valued functions on the phase space with the physical parameters of a statistical theory, in such a way that the statistical theorems are recovered. Condition (2.2) imposes a structure on the set of physical parameters suggested by their interpretation as physical attributes of objects. However, it is just this interpretation which is incompatible with the representation of the physical parameters by self-adjoint Hilbert space operators, and which is also rejected—implicitly or explicitly—by existing so-called 'hidden variable' theories.†

† See Section 6, below.

To sum up: a hidden variable extension of the quantum theory, as usually understood, involves the interpretation of the self-adjoint Hilbert space operators as representing the physical attributes of objects. The problem is then to embed the quantum theory into a more fundamental theory, so that these attributes are represented by real-valued functions on a phase space, in such a way that certain statistical conditions are satisfied. This means that commuting operators are represented by related functions, but since commutativity of operators is not transitive, conditions are imposed on the representatives of *non-commuting* operators which go beyond anything deducible from the quantum theory. Although this embedding can be shown to be impossible, this result is quite irrelevant to those theories which have actually been proposed as 'hidden variable' theories.†

The question of a hidden variable extension of the quantum theory has been further confused because the proofs of von Neumann, and Jauch and Piron, are actually much weaker results than the Kochen and Specker proof, and do not even establish the impossibility of the embedding. These proofs will be discussed below.

3. *von Neumann's Proof*

Kochen and Specker have shown (Kochen & Specker, 1967c) that von Neumann's proof amounts to a demonstration of the non-existence of a function $\mathrm{Av}: \mathscr{Q} \to \mathscr{R}$, which maps the set $\mathscr{Q}$ of self-adjoint Hilbert space operators into the set $\mathscr{R}$ of real numbers, satisfying the conditions:

$$\begin{array}{lll}
\text{(i)} & \mathrm{Av}(\mathbf{I}) = 1 & \\
\text{(ii)} & \mathrm{Av}(k\mathbf{A}) = k\,\mathrm{Av}(\mathbf{A}) & \text{for all } k \in \mathscr{R}, \mathbf{A} \in \mathscr{Q} \\
\text{(iii)} & \mathrm{Av}(\mathbf{A}^2) = \mathrm{Av}^2(\mathbf{A}) & \text{for all } \mathbf{A} \in \mathscr{Q} \\
\text{(iv)} & \mathrm{Av}(\mathbf{A}+\mathbf{B}) = \mathrm{Av}(\mathbf{A}) + \mathrm{Av}(\mathbf{B}) & \text{for all } \mathbf{A}, \mathbf{B} \in \mathscr{Q}
\end{array} \tag{3.1}$$

† Bell (1966), first pointed out a similar feature of Gleason's theorem. It follows as a corollary to Gleason's theorem that the average values of physical parameters represented by commuting operators cannot be additive for dispersion-free states. This is a stronger result than von Neumann's theorem, in which the non-existence of hidden variables is proved on the basis of an additivity requirement for the average values of all physical parameters, including those represented by non-commuting operators. (See Section 3, below.) Bell argued that the statistical relations of the quantum theory can be recovered from a 'hidden variable theory' in which Gleason's theorem is false for dispersion-free states. This is certainly true, but such a theory would not be a hidden variable theory in the sense of an embedding satisfying conditions (I), (II), and (III), because condition (2.2) would not be satisfied. The proof of Kochen and Specker (or the corollary to Gleason's theorem) is not inadequate *merely* because the assumptions are too restrictive. The assumptions in fact define the problem of proving the impossibility of a hidden variable extension of the quantum theory in the usual sense, and without these assumptions there is no problem (i.e., any proof would be quite arbitrary, and therefore uninteresting). These remarks do not apply to the assumptions of von Neumann, or Jauch and Piron. (See Sections 3 and 4, below.)

Av(**A**) is the average value of the physical parameter represented in the theory by the operator **A**. Condition (iii) therefore defines what von Neumann calls a 'dispersion-free' statistical state. According to von Neumann, if a hidden variable extension of the quantum theory is possible, then there exists a function Av for each set of values of the hidden variables (i.e., for each point in the phase space of hidden states) satisfying conditions (i), (ii), (iii), and (iv).

Kochen and Specker have pointed out that the proof of the non-existence of such a function is trivial. It follows from the above conditions that the function Av is multiplicative for commuting operators, i.e., that $\mathrm{Av}(\mathbf{AB}) = \mathrm{Av}(\mathbf{A})\mathrm{Av}(\mathbf{B})$ if $\mathbf{AB} = \mathbf{BA}$. From this it is easy to prove that Av(**A**) must be an eigenvalue of **A**. Hence condition (iv) cannot be satisfied for non-commuting operators, because the eigenvalues of non-commuting operators are not in general additive.

The Kochen and Specker proof of the impossibility of a hidden variable extension of the quantum theory involves the non-existence of a real-valued function on the set of physical parameters (represented by self-adjoint Hilbert space operators) which is multiplicative and linear *only on coherent parameters*. This is the homomorphism $h: \mathscr{Q} \to \mathscr{R}$, referred to in Section 2. von Neumann's proof is much weaker and establishes only the non-existence of a real-valued function which is multiplicative on coherent physical parameters and *linear on all parameters*. Hence, the proof excludes only a relatively uninteresting class of embeddings, namely those embeddings characterized by a map h satisfying the conditions:

$$\begin{aligned} &h(AB) = h(A)\,h(B), \text{ for coherent } A, B \\ &h(k_1 A + k_2 B) = k_1 h(A) + k_2 h(B), \text{ for all } A, B \end{aligned} \tag{3.2}$$

Since $h_\lambda(A) = f_A(\lambda)$, this means that the value associated with a linear combination of two non-coherent physical parameters, for a particular hidden state λ, is equal to the linear combination of the values associated with the non-coherent parameters. Evidently this is an unreasonable restriction on a hidden variable extension of the quantum theory. In other words, von Neumann's proof does not rule out a large class of hidden variable extensions of the quantum theory, i.e., theories characterized by a phase space $\mathscr{X}$ of hidden states, a probability measure ρ_ψ on $\mathscr{X}$ for each statistical state ψ, and a real-valued function f_A on $\mathscr{X}$ satisfying condition (2.2) associated with each physical parameter A, such that all the statistical theorems of quantum kinematics are recovered.

von Neumann's own proof was much more complicated than the reformulation by Kochen and Specker, and it seems reasonable to suppose that he did not realize its inadequacy. On the basis of assumptions (3.1) and an additional assumption (that $\mathrm{Av}(\mathbf{A}) \geqslant 0$ if **A** is 'by nature' non-negative, e.g. if **A** is expressible as the square of another operator), von Neumann demonstrated that there exists a linear, semi-definite, self-adjoint operator **U**, such that for any operator **A**:

$$\mathrm{Av}(\mathbf{A}) = \mathrm{Tr}(\mathbf{UA}) \tag{3.3}$$

Hence, every statistical state satisfying (3.1) can be associated with a certain statistical operator from which the average value of any physical parameter can be deduced according to a certain algorithm. Since there is no physically meaningful statistical operator which gives zero dispersion for all physical parameters, von Neumann's first conclusion was that there are no dispersion-free states. In addition, there are statistical operators which represent 'pure' or 'homogeneous' states, i.e., statistical states which cannot be expressed as a probability distribution of different states. The statistics of the pure state cannot result from averaging over dispersion-free hidden states because, firstly, the pure state could then be represented as a mixture of two different statistical states and, secondly, because the dispersion-free hidden states do not exist.

This argument is, of course, much more involved than the Kochen and Specker reformulation, and because of its redundancy the significance of condition (iv) is not clear.† It is not sufficient to prove that dispersion-free states do not exist in a theory which has some of the essential characteristics of the quantum theory. This still leaves open the possibility that the quantum theory can be embedded into a hidden variable theory in the usual sense, i.e., that there exists a homomorphism from the set of physical parameters of the quantum theory into the set of real-valued functions on a suitable phase space, in such a way that various statistical conditions are satisfied. The homomorphism is a function h satisfying the conditions:

$$\begin{aligned} h(AB) &= h(A)\,h(B) \\ h(k_1 A + k_2 B) &= k_1 h(A) + k_2 h(B) \end{aligned} \tag{3.4}$$

for coherent physical parameters. Any conditions on the function h for non-coherent physical parameters, such as that imposed by von Neumann in condition (3.2) or (iv), restricts the class of hidden variable extensions of the quantum theory in an arbitrary manner.

4. *The Proof of Jauch and Piron*

Jauch and Piron attempt to prove the impossibility of a hidden variable extension of the quantum theory by investigating the order relations of quantum theoretical *propositions*, i.e., the 'logic' of the quantum theory.‡

From the point of view of Kochen and Specker, the set of physical

† This is von Neumann's condition B′. [See von Neumann (1955, p. 311).] Bell (1966), first pointed out the inadequacy of this condition. He did not, however, make any distinction between his criticism of von Neumann's proof and his criticism of an application of Gleason's theorem to the question of a hidden variable extension of the quantum theory. (See footnote on p. 7.)

‡ This concept of a 'quantum logic' was first introduced by Birkhoff, G. & von Neumann, J. (1936). *Annals of Mathematics*, 823.

parameters of a theory forms a partial algebra over the field $\mathscr{R}$ of real numbers, with respect to the relation of coherence. If A is an idempotent element of the set $\mathscr{P}$ (i.e., $A^2 = A$), then the possible values of A can only be 1 or 0 (if it is assumed that the possible values of A^2 are the squares of the corresponding values of A). Each such idempotent physical parameter may be associated with a *proposition* of the theory, the two possible values corresponding to truth and falsity. The set of propositions of a physical theory therefore forms a *partial Boolean algebra*, which may be defined as a set of elements with Boolean operations which satisfy the Boolean axioms for coherent elements. Evidently, the set of idempotent elements of a partial algebra forms a partial Boolean algebra.

The physical parameters $(A, B, \ldots)$ of the quantum theory are represented by the self-adjoint operators $(\mathbf{A}, \mathbf{B}, \ldots)$ on a complex Hilbert space, which form a partial algebra over the field $\mathscr{R}$ of reals, with respect to the relation of commutativity. The idempotent self-adjoint operators are projection operators $(\mathbf{P}_a, \mathbf{P}_b, \ldots)$, and form a partial Boolean algebra. This partial Boolean algebra is isomorphic to the partial Boolean algebra of closed linear subspaces $(\mathscr{V}_a, \mathscr{V}_b, \ldots)$ of the Hilbert space, because every projection operator corresponds uniquely to a closed linear subspace. It is also isomorphic to the partial Boolean algebra of quantum theoretical propositions $(a, b, \ldots)$.

The Boolean operations $\vee$, $\wedge$, and $'$ (join, meet, and complement, analogous to the set-theoretical operations union, intersection, and complement) for the set of propositions are related to the ring operations $+$, $.$, and $-$, for the set of projection operators, as follows: a' is associated with the projection operator $\mathbf{I} - \mathbf{P}_a$ (which defines the subspace $\mathscr{H} - \mathscr{V}_a$, where $\mathscr{H}$ is the whole Hilbert space). The proposition $a \wedge b$ is associated with the projection operator $P_a . P_b$ (which defines the sub-space consisting of those Hilbert space vectors which belong to both $\mathscr{V}_a$ and $\mathscr{V}_b$). The proposition $a \vee b$ is associated with the projection operator

$$\mathbf{I} - (\mathbf{I} - \mathbf{P}_a)(\mathbf{I} - \mathbf{P}_b) = \mathbf{P}_a + \mathbf{P}_b - \mathbf{P}_a . \mathbf{P}_b$$

(which defines the subspace spanned by those Hilbert space vectors which are linear combinations of vectors from $\mathscr{V}_a$ and vectors from $\mathscr{V}_b$). (These operations are so far assumed to be defined only for commuting projection operators and their corresponding coherent idempotent physical parameters.)

A binary relation $\leqslant$ can be introduced into a partial Boolean algebra, defined by:

$$a \leqslant b \quad \text{if and only if} \quad a \wedge b = a \tag{4.1}$$

It follows from the axioms of a partial Boolean algebra that the relation $\leqslant$ defines a partial order, i.e., it is reflexive ($a \leqslant a$), antisymmetric (if $a \leqslant b$ and $b \leqslant a$, then $a = b$), and transitive (if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$). The sentential connectives *and*, *or*, and *not* have similar properties to the Boolean operations $\wedge$, $\vee$, and $'$ (which are referred to as conjunction, disjunction, and negation in the logical terminology). This suggests introducing an operation '$\Rightarrow$' analogous to logical implication:

$$a \Rightarrow b = a' \vee b \tag{4.2}$$

It follows that:

$$a \leqslant b \quad \text{if and only if} \quad a \Rightarrow b = 1 \tag{4.3}$$

A *lattice* is a partially ordered set in which every pair of elements has both a supremum (or least upper bound), and an infimum (or greatest lower bound). The supremum and infimum are denoted by $a \vee b$ and $a \wedge b$, respectively. The elements of a distributive lattice satisfy the distributive laws

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad \text{and} \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

A lattice is complemented if for every element a there exists a complement a', defined by $a \wedge a' = 0$, and $a \vee a' = 1$, where 0 and 1 are two distinguished elements such that $0 \leqslant a$ and $a \leqslant 1$ for every a It is a theorem that a complemented distributive lattice is a Boolean algebra. The complement of an element, as defined here, is not necessarily unique. By analogy with the concept of orthogonal subspaces in a vector space, a unique *ortho-complement*, $a^{\perp}$, of a lattice element, a, can be defined, so that $(a^{\perp})^{\perp} = a$, $a \wedge a^{\perp} = 0$, $a \vee a^{\perp} = 1$, and $a \leqslant b$ implies $b^{\perp} \leqslant a^{\perp}$. In a distributive lattice, complementation is unique and is equivalent to ortho-complementation.†

Since the set of propositions of a physical theory forms a partial Boolean algebra, an order relation can be defined so that the set of propositions becomes a partially ordered system (i.e., the relation is reflexive, anti-symmetric, and transitive). The system will not necessarily be a lattice, because the Boolean operations are only defined for propositions corresponding to coherent physical parameters, and hence if some parameters are not coherent, not every pair of elements in the system will have a supremum and an infimum. However, the Boolean operations can simply be extended to all propositions. For example, in the quantum theory, if the Hilbert space operators $\mathbf{P}_a$ and $\mathbf{P}_b$ do not commute, then the proposition $a \wedge b$ can be associated with the subspace consisting of those Hilbert space vectors which belong to both $\mathscr{V}_a$ and $\mathscr{V}_b$ ($\mathscr{V}_a$ and $\mathscr{V}_b$ being the closed linear subspaces corresponding to the projection operators $\mathbf{P}_a$ and $\mathbf{P}_b$ respectively). Similarly, the proposition $a \vee b$ can be associated with the subspace spanned by those Hilbert space vectors which are linear combinations of vectors from $\mathscr{V}_a$ and vectors from $\mathscr{V}_b$. (In this case, $\mathbf{P}_a . \mathbf{P}_b$ and $\mathbf{P}_a + \mathbf{P}_b - \mathbf{P}_a . \mathbf{P}_b$ are not projection operators. The projection operators corresponding to $a \wedge b$ and $a \vee b$ have to be defined by a limiting process—$a \wedge b$ corresponds to the projection operator: $\lim_{n\to\infty}(\mathbf{P}_a . \mathbf{P}_b)^n$.) In fact, the partial Boolean algebra may be regarded as a family of Boolean algebras,

† See, for example, Birkhoff, G. (1940). *Lattice Theory*, Theorem 6.1, p. 88. American Mathematical Society, New York.

each of which contains elements which are non-coherent with some elements of all the other Boolean algebras in the family. The extension of the Boolean operations to non-coherent elements belonging to different Boolean algebras in this family will not, of course, convert the partial Boolean algebra into a Boolean algebra (unless it can be embedded into a Boolean algebra). But it will convert the partial Boolean algebra, considered as a partially ordered set, into a complemented lattice. In view of the theorem that a complemented distributive lattice is a Boolean algebra, this lattice cannot be distributive unless the partial Boolean algebra is embeddable into a Boolean algebra.

Jauch and Piron prefer to analyse the ortho-complemented lattice of quantum theoretical propositions rather than the corresponding partial Boolean algebra. They introduce the concept of *compatibility*, analogous to the concept of coherence: two propositions a and b are compatible if the sub-lattice generated by taking infima and ortho-complements is isomorphic to a complemented, distributive lattice, i.e., to a Boolean algebra.† Since the partial Boolean algebra of quantum theoretical propositions cannot be embedded into a Boolean algebra, not all quantum theoretical propositions are compatible. In other words, the lattice of quantum theoretical propositions cannot be distributive. As a simple example, consider the propositions a, b, and $b^{\perp}$ associated with the non-commuting projection operators P_a, P_b, and $P_b{}^{\perp}$, such that $\mathscr{V}_{a \wedge b} = \mathscr{V}_{a \wedge b^{\perp}} = \phi$ (the null subspace). Then $a \wedge (b \vee b^{\perp}) = a \wedge I = a$, but $(a \wedge b) \vee (a \wedge b^{\perp}) = 0$.‡

The propositions of a hidden variable theory are all compatible by definition, because they are the idempotent elements of a commutative algebra (the set $\mathscr{R}^{\mathscr{X}}$ of all functions $f: \mathscr{X} \to \mathscr{R}$ from the phase space of hidden states into the reals), and hence form a Boolean lattice. The result of Kochen and Specker suggests that it should be possible to provide a direct lattice theoretical proof of the impossibility of embedding the lattice of quantum theoretical propositions into a Boolean lattice, i.e., of developing a hidden variable extension of the quantum theory in such a way that all propositions become compatible. This should follow from the specific order structure of the lattice of quantum theoretical propositions (some non-Boolean lattices can be embedded into a Boolean lattice).

In fact, Jauch and Piron do not attempt to prove the impossibility of this embedding. What they do prove is that there can be no dispersion-free states on a lattice in which some propositions are incompatible (Jauch & Piron, 1963). But this result is subject to the same objection as von Neumann's result. There is still the possibility that a homomorphism exists from the lattice of quantum theoretical propositions into a Boolean lattice, which implies the existence of a function $h': \mathscr{L} \to \{0, 1\}$, mapping

† A different definition of compatibility is proposed in the paper by Jauch & Piron (1963). See Piron's thesis [Piron, C. (1964). *Helvetica physica acta*, **37**, 439] for the proof of the equivalence of various definitions of compatibility.

‡ Such an example was first proposed by Birkhoff, G. & von Neumann, J. (1936). *Annals of Mathematics*, **37**, 823.

the lattice of quantum theoretical propositions onto $\{0,1\}$, under the conditions:

$$\begin{aligned} h'(a \wedge b) &= h'(a)\,h'(b) \\ h'(a \vee b) &= h'(a) + h'(b) - h'(a)\,h'(b) \end{aligned} \tag{4.4}$$

for all *compatible* $a, b \in \mathscr{L}$.†

Jauch and Piron define a state as an additive mapping ω from $\mathscr{L}$ into the interval $[0,1]$, such that $\omega(0) = 0$ and $\omega(1) = 1$. It is also required that, for any $a, b \in \mathscr{R}$, $\omega(a \wedge b) = 1$ if $\omega(a) = \omega(b) = 1$. Hence a proof of the non-existence of dispersion-free states on a certain lattice $\mathscr{L}$ amounts to a proof of the non-existence of a function $\omega: \mathscr{L} \to \{0,1\}$, satisfying the conditions:

$$\omega(a \vee b) + \omega(a \wedge b) = \omega(a) + \omega(b), \text{ for all compatible } a, b \in \mathscr{L} \tag{4.5}$$

$$\omega(a \wedge b) = 1, \text{ if } \omega(a) = \omega(b) = 1, \text{ for any } a, b \in \mathscr{L} \tag{4.6}$$

Condition (4.5) follows from the assumption of additivity for compatible propositions and is numbered (3) in the paper by Jauch and Piron. Condition (4.6) is numbered (4)°. Together these conditions imply that ω satisfies conditions (4.4) for compatible propositions.‡ The requirement that (4)° holds for incompatible propositions as well arbitrarily restricts the class of hidden variable extensions of the quantum theory.

The conclusion of Jauch and Piron cannot, therefore, be regarded as a significant improvement on von Neumann's result.§ The proof does not rule out a large class of hidden variable extensions of the quantum theory characterized by a phase space $\mathscr{X}$ of hidden states, a probability measure ρ_ψ on $\mathscr{X}$ for each statistical state ψ, and a real-valued function f_A on $\mathscr{X}$ satisfying conditions (2.2) associated with each physical parameter A, such that all the statistical theorems of quantum kinematics are recovered. Condition (2.2), i.e., $f_A(\lambda) = \alpha$ implies $f_{g(A)}(\lambda) = g(\alpha)$, requires that $f_{A+B}(\lambda) = \alpha + \beta = f_A(\lambda) + f_B(\lambda)$ and $f_{AB}(\lambda) = \alpha\beta = f_A(\lambda)\,f_B(\lambda)$, if $f_A(\lambda) = \alpha$, $f_B(\lambda) = \beta$ and A, B are coherent physical parameters. This means that each

† The map $h': \mathscr{L} \to \{0,1\}$ corresponds to the homomorphism $h: \mathscr{B} \to \{0,1\}$ from the partial Boolean algebra of idempotent physical parameters represented by projection operators in Hilbert space onto $\{0,1\}$. The homomorphism h satisfies the conditions:

$$h(\mathbf{P}_a . \mathbf{P}_b) = h(\mathbf{P}_a)h(\mathbf{P}_b)$$

and

$$h(\mathbf{P}_a + \mathbf{P}_b) = h(\mathbf{P}_a) + h(\mathbf{P}_b), \quad \text{for } \textit{commuting } \mathbf{P}_a, \mathbf{P}_b$$

The proposition $a \wedge b$ is associated with the projection operator $\mathbf{P}_a . \mathbf{P}_b$. The proposition $a \vee b$ is associated with the projection operator $\mathbf{P}_a + \mathbf{P}_b - \mathbf{P}_a . \mathbf{P}_b$.

‡ It suffices to prove that (4.5) and (4.6) imply that $\omega(a \wedge b) = \omega(a)\,\omega(b)$, for compatible a, b. From the definition of compatibility it follows that $a \vee b$ can be expressed in terms of disjoint elements: $a \vee b = (a \wedge b^\perp) \vee (a \wedge b) \vee (b \wedge a^\perp)$. Hence (4.5) can be written as: $2\omega(a \wedge b) = \omega(a) + \omega(b) - \omega(a \wedge b^\perp) - \omega(b \wedge a^\perp)$. From this expression it is easy to prove a contradiction if $\omega(a \wedge b) \neq \omega(a)\,\omega(b)$.

§ Bell (1966), first pointed out the inadequacy of condition (4)°. He did not, however, distinguish between his objection to condition (4)° and his objection to the use of a corollary to Gleason's theorem in this context. (See footnote on p. 7.)

hidden state $\lambda \in \mathscr{X}$ defines a homomorphism h_λ: $\mathscr{P} \to \mathscr{R}$ of the partial algebra $\mathscr{P}$ of quantum theoretical physical parameters into $\mathscr{R}$, defined by $h_\lambda(A) = f_A(\lambda)$. If A, B are coherent idempotent physical parameters, then the corresponding compatible propositions a, b satisfy conditions (4.4): $h_\lambda'(a \wedge b) = h_\lambda'(a)h_\lambda'(b)$, $h_\lambda'(a \vee b) = h_\lambda'(a) + h_\lambda'(b) - h_\lambda'(a)h_\lambda'(b)$. It is possible to develop a hidden variable extension of the quantum theory in which (4.4) is satisfied for compatible propositions, without necessarily satisfying condition (4.6) or (4)° for all propositions, as required by Jauch and Piron.

It might be argued that the Jauch and Piron proof is successful as an attempt to provide a lattice theoretical analogue of von Neumann's proof, which utilized the properties of Hilbert space. However, von Neumann's result is manifestly inadequate, since it does not exclude a large class of hidden variable extensions of the Hilbert space quantum theory. What is required is a lattice theoretical analogue of the Kochen and Specker proof (or Gleason's theorem)—since it is this result which is the 'strongest' hidden variable theorem for Hilbert space quantum theory. If condition (4.6) or (4)° is both necessary and sufficient† for the required conclusion with respect to the general orthomodular lattice considered by Jauch and Piron, this indicates only that the proposed lattice structure is much more general than the quantum theory. In order to provide a direct lattice theoretical proof of the impossibility of developing a hidden variable extension of the quantum theory, i.e., of embedding the lattice of quantum theoretical propositions into a Boolean algebra, an additional defining characteristic is probably required for the lattice of quantum theoretical propositions, which together with the other axioms of Jauch and Piron will entail the non-existence of a function $h' : \mathscr{L} \to \{0, 1\}$ satisfying conditions (4.4).

5. *The Proof of Margenau and Cohen*

In 1949, Moyal published a classic paper (Moyal, 1949a) in which he investigated the possibility of reproducing the quantum statistics from joint probability distributions for position and momentum, i.e., of relating the statistical states of the quantum theory to probability distributions over a classical phase space $\mathscr{X}$, where the points $\lambda \in \mathscr{X}$ are the values of the position and momentum variables. Moyal pointed out that if a rule is specified for associating functions of physical parameters A, B—represented by non-commuting operators **A**, **B** in the quantum theory—with corresponding functions of these operators, then the moments are determined by the rule, and hence the distribution function $F(A, B)$ is determined. Thus, there is a close relationship between a distribution function and a 'correspondence rule'. Moyal developed his theory for a correspondence

† This has been suggested by Jauch and Piron. [See Jauch, J. M. and Piron, C. (1968). *Reviews of Modern Physics*, **40**, 228.]

rule which reduces to Weyl's rule† for position and momentum variables and an associated distribution function first introduced by Wigner.‡

This approach has recently been extended by Margenau & Cohen (1967a), who have proposed a general expression for normalized, real-valued distribution functions on the phase space of position and momentum variables satisfying the conditions that integration over the momentum yields the quantum theoretical probability distribution for position, and integration over the position yields the quantum theoretical probability distribution for momentum. This expression is then related to a general expression for a correspondence rule. It is claimed that these expressions generate the totality of suitable distribution functions on the phase space of position and momentum variables, and the totality of correspondence rules. Thus, each appropriate distribution function can be associated with a certain correspondence rule in an explicit way. The Wigner distribution and the Weyl rule are special cases of these general expressions.

Margenau and Cohen are therefore in a position to answer the following question: Is it possible to find a normalized, real-valued distribution function, F, on the phase space of position and momentum variables, and an associated correspondence rule, such that for any real-valued function of position and momentum, f_A, and its corresponding operator **A**:

$$\mathrm{Av}_{\psi}(\mathbf{A}) = \int_{\mathscr{X}} f_A(\lambda)\, F_{\psi}(\lambda)\, d\lambda \tag{5.1}$$

and also:

$$\mathrm{Av}_{\psi}(g(\mathbf{A})) = \int_{\mathscr{X}} g(f_A(\lambda))\, F_{\psi}(\lambda)\, d\lambda \tag{5.2}$$

for any function g. Margenau and Cohen refer to Cohen's thesis for the proof that the same distribution function F cannot be used to calculate the average values of both **A** and g(**A**). This is equivalent to the statement that the same correspondence rule cannot be used to derive both **A** from f_A and g(**A**) from $g(f_A)$.

Although Margenau and Cohen do not attempt to relate their conclusion to the proofs of von Neumann, *et al.*, what they have demonstrated amounts

† Weyl's rule is:

$$q^n p^m \rightarrow \frac{1}{2^n} \sum_{l=0}^{n} \binom{n}{l} \boldsymbol{q}^{n-l} \boldsymbol{p}^m \boldsymbol{q}^l$$

See Moyal, J. E. (1949). *Proceedings of the Cambridge Philosophical Society. Mathematical and Physical Sciences*, **45**, 104.

‡ The Wigner distribution function is:

$$F(q, p) = \frac{1}{2\pi} \int \psi^* \left(q - \frac{1}{2h\tau}\right) \exp(-i\tau p)\, \psi \left(q + \frac{1}{2h\tau}\right) d\tau$$

See Moyal, J. E. (1949). *Proceedings of the Cambridge Philosophical Society. Mathematical and Physical Sciences*, **45**, 102.

to a similar result, restricted to the phase space of position and momentum variables. In the notation of Margenau and Cohen, the equations (5.1) and (5.2) are expressed as (Margenau & Cohen, 1967b):

$$\langle\psi|G(\mathbf{q},\mathbf{p})|\psi\rangle = \iint g(q,p)\,F(q,p)\,dq\,dp \tag{5.3}$$

$$\langle\psi|K(G(\mathbf{q},\mathbf{p}))|\psi\rangle = \iint K(g(q,p))\,F(q,p)\,dq\,dp \tag{5.4}$$

where $\mathbf{q}$, $\mathbf{p}$ are the operators corresponding to the position and momentum parameters q, p, which are also the variables of the phase space. The condition (2.2) of Kochen and Specker is expressed as (Margenau & Cohen, 1967c):

$$\begin{aligned} g(q,p) &\to G(\mathbf{q},\mathbf{p}) \\ K(g(q,p)) &\to K(G(\mathbf{q},\mathbf{p})) \end{aligned} \tag{5.5}$$

The analysis of Margenau and Cohen is interesting, because it can be directly compared with Moyal's theory. Moyal realized the impossibility of embedding the partial algebra of physical parameters of the quantum theory into the commutative algebra of real-valued functions on the classical phase space of position and momentum variables in such a way that conditions (5.1) and (5.2) are satisfied. He cited the example of a harmonic oscillator, whose energy eigenvalues form a discrete set $E_n = (n+\frac{1}{2})h\nu$. The corresponding eigenfunctions are continuous functions of q, or continuous functions of p (Hermite functions). Hence a joint distribution for p and q must extend continuously over the whole p, q plane, while a joint distribution for the energy and phase angle (certain functions of p and q) will be concentrated on a set of ellipses defined by the discrete energy eigenvalues. Moyal (1949b) concluded:

> 'We are thus forced to the conclusion that *phase-space distributions are not unique for a given state, but depend on the variables one is going to measure*.... The statistical interpretation of quantum kinematics will thus have to give methods for setting up the appropriate phase-space distribution of each *basic system of dynamical variables* in terms of the wave vectors, and for transforming such distributions into one another.'

It follows immediately from the Kochen and Specker definition of coherence that a coherent set of physical parameters can be embedded into the commutative algebra of real-valued functions on a certain phase space. If every member A_i of this coherent set is expressed as a function α_i of the physical parameter A, then the phase space $\mathscr{X}_A$ can be defined as the set of possible values of A, and each A_i can be associated with the function $\alpha_i: \mathscr{X}_A \to \mathscr{R}$. It is easy to see that if a probability measure $\rho_\psi^{(A)}$ is defined on $\mathscr{X}_A$, satisfying the condition:

$$\mu_{\psi A}(\mathscr{M}) = \rho_\psi^{(A)}(f_A^{-1}(\mathscr{M})) \tag{5.6}$$

then, for any $A_i = \alpha_i(A)$:

$$\mu_{\psi A_i}(\mathscr{M}) = \rho_\psi^{(A)}(f_{A_i}(\mathscr{M})) \tag{5.7}$$

If the physical parameters of a statistical theory are not all coherent, a phase space $\mathscr{X}_A$, $\mathscr{X}_B$,...can be defined for each coherent set $\mathscr{A}$, $\mathscr{B}$,..., so that appropriate relative measures $\rho_\psi^{(A)}$, $\rho_\psi^{(B)}$,...satisfy conditions (5.6) and (5.7). Moyal constructed his phase space, $\mathscr{X}_{RS}$, out of the possible values of appropriate non-coherent pairs of physical parameters. He proved the theorem (Moyal, 1949c) that the Weyl correspondence rule can be applied in a consistent way to any function $g(k_1 R + k_2 S)$ of a linear combination of the basic parameters R, S. Hence, if the physical parameter G represented by the operator $\mathbf{G}$ is relevant, the phase space is defined as $\mathscr{X}_{RS}$, for appropriate physical parameters R, S such that G corresponds to the real-valued function $g(k_1 R + k_2 S)$ on $\mathscr{X}_{RS}$. Moyal's theory may therefore be regarded as an expression of the essential content of the proof of Kochen and Specker.

6. *The 'Hidden Variable' Theories of Bohm and his Collaborators*

In Section 2, a hidden variable extension of a statistical theory was characterized as an embedding of the partial algebra of physical parameters into the commutative algebra of real-valued functions on a suitable phase space of hidden states, so that the probability measures $\mu_{\psi A}$ (for a fixed $\psi \in \mathscr{S}$, and all $A \in \mathscr{P}$) are derived from a probability measure ρ_ψ defined on the phase space. Moyal's theory does not conform to this definition, because the phase space probability distribution associated with a certain statistical state ψ is not fixed but varies with the relevant physical parameters, i.e., the probability distributions may be said to be 'relative to the measurement context'.

Bohm also explicitly pointed out this aspect of his original 1951 'hidden variable' theory. In a section on von Neumann's proof, he wrote (Bohm, 1952b):

> 'His conclusions are subject, however, to the criticism that in his proof he has implicitly restricted himself to an excessively narrow class of hidden parameters and in this way has excluded from consideration precisely those types of hidden parameters which have been proposed in this paper.... For example, if we consider two noncommuting observables, p and q, then von Neumann shows that it would be inconsistent with the usual rules of calculating quantum-mechanical probabilities to assume that there were in the observed system a set of hidden parameters which simultaneously determined the results of measurement of position and momentum "observables". *With this conclusion we are in agreement.* However, in our suggested new interpretation of the theory, the so-called "observables" are...not properties belonging to the observed system alone, but instead potentialities whose precise development depends just as much on the observing apparatus as on the observed system.... *Thus, the statistical distribution of "hidden" parameters to be used in calculating averages in a momentum measurement*

is different from the distribution to be used in calculating averages in a position measurement. von Neumann's proof...that no single distribution of hidden parameters could be consistent with the results of the quantum theory is therefore irrelevant here, since in our interpretation of measurements of the type that can now be carried out, the distribution of hidden parameters varies in accordance with the different mutually exclusive experimental arrangements of matter that must be used in making different kinds of measurements. In this point, we are in agreement with Bohr, who repeatedly stresses the fundamental role of the measuring apparatus as an inseparable part of the observed system.' (Italics inserted.)

Bohm's theory may be regarded as an attempt to articulate the fundamental confusion in the approach of Kochen and Specker (or the von Neumann approach) to the hidden variable problem, i.e., the interpretation of the physical parameters represented in the theory by self-adjoint Hilbert space operators as attributes of 'quantum objects', and the consequent rejection of hidden variable theories. The Copenhagen interpretation of the quantum theory, as expressed by Bohr, is motivated by the same insight. Bohr repeatedly emphasized the 'impossibility of any sharp separation between the behaviour of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear' (Bohr, 1949a). Hence, he proposed that 'evidence obtained under different experimental conditions cannot be comprehended within a single picture, but must be regarded as *complementary* in the sense that only the totality of the phenomena exhausts the possible information about the objects' (Bohr, 1949a). This is another way of expressing the fact that a conventional phase space or hidden variable interpretation of the quantum statistics, as discussed by von Neumann and Kochen and Specker will involve *relative phase spaces* and *relative probability measures*, i.e., different phase spaces and associated probability measures *for each coherent set of physical parameters*. The Kochen and Specker theorem that the different phase spaces and associated probability measures for non-coherent sets of physical parameters cannot be reduced to a single phase space and associated probability measure is the content of Bohr's principle of complementarity.

In the theory of Bohm & Bub (1966), a Hilbert space, $\mathscr{H}_\xi$, is introduced in addition to the Hilbert space $\mathscr{H}_\psi$ of 'statistical states' of the quantum theory. The equation of motion proposed for ψ involves ξ and describes a process in which ψ is projected onto a particular eigenvector of the representation in which the matrix of the operator representing a certain relevant physical parameter is diagonal. It follows from the form of this equation that the resulting eigenvector is determined by the greatest ratio $|\psi_i|^2/|\xi_i|^2$, where ψ_i and ξ_i $(i=1,\ldots,n)$ are the components of ψ and ξ respectively in the relevant representation (say, the A-representation), assuming for simplicity that $\mathscr{H}_\psi$ and $\mathscr{H}_\xi$ are n-dimensional Hilbert spaces and $\mathbf{A}$ is non-degenerate. Hence, it might be supposed that the theory is simply a straightforward counter-example to the Kochen and Specker proof, with a phase space $\mathscr{X} = \mathscr{H}_\psi \times \mathscr{H}_\xi$, real-valued functions f_A on $\mathscr{X}$ defined by the algorithm:

$$f_A(\psi, \xi) = A_k \quad \text{if} \quad \frac{|\psi_k|^2}{|\xi_k|^2} > \frac{|\psi_i|^2}{|\xi_i|^2} \quad \text{for all } i \neq k \tag{6.1}$$

and a probability measure ρ_ψ on $\mathscr{X}$ defined as the product of an atomic probability measure on $\mathscr{H}_\psi$ concentrated at the point ψ (where ψ is the relevant 'statistical state') and a normalized, uniform distribution on the surface of the unit hypersphere in $\mathscr{H}_\xi$.

In the n-dimensional case, it is not difficult to prove that $\mu_{\psi A}(\mathscr{M}) = \rho_\psi(f_A^{-1}(\mathscr{M}))$, if $\mathbf{A}$ is non-degenerate. Suppose, for simplicity, that the only eigenvalue of $\mathbf{A}$ in the set $\mathscr{M}$ is A_k, which can be re-labelled A_1 without loss of generality. Then, according to the quantum theory, $\mu_{\psi A}(\mathscr{M}) = |\psi_1|^2$. The inequalities (6.1), defining the set $f_A^{-1}(\mathscr{M})$ in the space $\mathscr{X}$ can be expressed as:

$$r_i^2 > \frac{|\psi_1|^2}{|\psi_i|^2} r_1^2 \qquad (i = 2, \ldots, n) \tag{6.2}$$

or:

$$\varepsilon_i^2(\phi_1, \ldots, \phi_{n-1}) > \frac{|\psi_i|^2}{|\psi_1|^2} \varepsilon_1^2(\phi_1, \ldots, \phi_{n-1}) \tag{6.3}$$

where $\xi_i = r_i \exp(i\theta_k)$, and the equations $r_i = r\varepsilon_i(\phi_1, \ldots, \phi_{n-1})$ define a transformation from the coordinates $r_1, \ldots, r_n$ to spherical coordinates $r, \phi_1, \ldots, \phi_{n-1}$.† For a fixed ψ, (6.3) is a set of $n-1$ inequalities defining limits on the $n-1$ angle variables $\phi_1, \ldots, \phi_{n-1}$ only. The measure of the set of points $f_A^{-1}(\mathscr{M})$ is computed by integrating a constant distribution function—$1/S = (n-1)!/2\pi^n$, where S is the surface area of the unit hypersphere in $\mathscr{H}_\xi$ defined by

$$\sum_{i=1}^{n} |\xi_i|^2 = 1$$

over the region on the surface of the unit hypersphere in $\mathscr{H}_\xi$ defined by the inequalities (6.2) or (6.3) for a fixed ψ, and integrating the resulting function of ψ over the space $\mathscr{H}_\psi$. The integration over $\mathscr{H}_\psi$ is trivial and yields the value of this function at the relevant point ψ; hence, the computation of $\rho_\psi(f_A^{-1}(\mathscr{M}))$ simply involves an integral over the variables $\phi_1, \ldots, \phi_{n-1}, \theta_1 \ldots \theta_n$, with the range of integration defined by (6.3) for a fixed ψ, specified by the relevant 'statistical state'.

† Explicitly: $r_1 = r\cos\phi_1$, $r_i = r\sin\phi_1 \ldots \sin\phi_{i-1}\cos\phi_i$, $r_n = r\sin\phi_1 \ldots \sin\phi_{n-1}$. The Jacobean of the transformation is:

$$J = \frac{\partial(r_1, \ldots, r_n)}{\partial(r, \phi_1, \ldots, \phi_{n-1})} = r^{n-1} \sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \ldots \sin\phi_{n-2}$$
$$= r^{n-1}\,\varepsilon(\phi_1, \ldots, \phi_{n-2})$$

so that:

$$dr_1 \ldots dr_n = r^{n-1}\, dr\, \varepsilon(\phi_1, \ldots, \phi_{n-1})\, d\phi_1 \ldots d\phi_{n-1}.$$

The element of volume in the space $\mathscr{H}_\xi$ is:

$$\begin{aligned} dV &= r_1 \ldots r_n \, dr_1 \ldots dr_n \, d\theta_1 \ldots d\theta_n \\ &= r^{2n-1} \, dr \, d\Phi \, d\theta_1 \ldots d\theta_n \end{aligned} \tag{6.4}$$

where $d\Phi$ involves only the $n-1$ angles $\phi_1, \ldots, \phi_{n-1}$. The surface element on the unit hypersphere is:

$$dS = d\Phi \, d\theta_1 \ldots d\theta_n \tag{6.5}$$

Let:

$$I = \int_0^\infty r^{2n-1} g(r^2) \, dr = \frac{(n-1)!}{2} \int_0^\infty \frac{G(t)}{t^n} \, dt \tag{6.6}$$

where $g(r^2)$ is any hyperspherically symmetrical function which leads to a convergent integral I, and g is the Laplace transform of G, i.e.,

$$g(r^2) = \int_0^\infty \exp(-r^2 t) \, G(t) \, dt$$

Then:

$$\begin{aligned} \rho_\psi(f_A^{-1}(\mathscr{M})) &= \frac{(2\pi)^n}{S} \int d\Phi \\ &= \frac{2^{n-1}(n-1)!}{I} \int d\Phi \int_0^\infty r^{2n-1} g(r^2) \, dr \\ &= \frac{2^{n-1}(n-1)!}{I} \int r_1 \ldots r_n g(r_1{}^2 + \cdots + r_n{}^2) \, dr_1 \ldots dr_n \\ &= \frac{(n-1)!}{2I} \int d(r_1{}^2) \ldots d(r_n{}^2) \int_{t=0}^\infty \exp[-(r_1{}^2 + \cdots + r_n{}^2) t] \, G(t) \, dt \\ &= \frac{(n-1)!}{2I} \int_{t=0}^\infty G(t) \, dt \int \exp[-(r_1{}^2 + \cdots + r_n{}^2) t] \, d(r_1{}^2) \ldots d(r_n{}^2) \\ &= |\psi_1|^2 \frac{(n-1)!}{2I} \int_0^\infty \frac{G(t)}{t^n} \, dt \\ &= |\psi_1|^2 \end{aligned} \tag{6.7}$$

The apparent simplicity of this interpretation of the theory is, however, illusory. In the present form of the theory, the functions f_A cannot be defined for degenerate operators **A**, such as the idempotent operators representing quantum theoretical propositions in a Hilbert space of more than two dimensions. The theory can obviously be extended in various ways to include degenerate operators,† but any such generalization will violate condition (2.2), i.e., the relationship $f_{g(A)} = g(f_A)$ will not be satisfied. In other words, the theory of Bohm and Bub is not a hidden variable extension of the quantum theory in the sense of the phase space theories considered by von Neumann, Jauch and Piron, and Kochen and Specker. In particular, the physical parameters of the quantum theory represented by self-adjoint Hilbert space operators are not interpreted as referring to physical attributes of 'quantum objects', and consequently the 'hidden variables' are not introduced as specifying 'hidden states' of 'quantum objects', probabilistically related to the 'statistical states' ψ.

Unfortunately, the emphasis on *measurement*, which has no place in a physical *theory*, has led to misunderstandings about the intention of the theories of Bohm and other workers in this field, and also of the Copenhagen interpretation. It is generally supposed that the change from classical to quantum theories has something to do with the impossibility of simultaneously measuring or observing certain physical attributes of micro-objects, since there is no similar measurability restriction for the corresponding physical attributes of macro-objects. On this view, classical mechanics is applicable to the macro-level, because physical attributes of macro-objects are represented in the theory by real-valued functions on the points (classical states) of a suitable vector space (phase space). Quantum mechanics is applicable to the micro-level, because physical attributes of micro-objects are represented in the theory by (the eigenvalues of self-adjoint) operators on a suitable vector space (Hilbert space). The non-commutativity of some operators then reflects the impossibility of simultaneously observing the corresponding physical attributes. This is proposed as the meaning of the uncertainty principle, or the feature of complementarity, and the reason for the essentially statistical character of the quantum theory. The hidden variable approach therefore seems to involve nothing more than the reactionary program to reinstate classical mechanics as the fundamental description of the micro-level, and the rejection of all that is novel in the quantum theory. It is claimed that hidden variables are introduced in order to characterize a phase space of micro-states on which real-valued functions can be defined to represent the physical attributes of micro-objects, in the manner of classical mechanics, so that the peculiar statistical relations of the quantum theory are simply explained by the incompleteness of the quantum theory. This concept of a hidden variable theory of quantum phenomena is indeed reactionary, and its consequences have been thoroughly explored by von Neumann *et al.* It has been shown that the algebraic structure of self-adjoint Hilbert space operators cannot

† For example, the Hilbert space $\mathscr{H}_\xi$ can be defined to have the same number of dimensions as the degeneracy of the relevant operator. This generalization is in fact required for the description of the Einstein–Podolski–Rosen paradox. Evidently, such a theory would be radically different from the hidden variable theories considered by von Neumann *et al.*

be embedded into the algebraic structure of real-valued phase space functions, so that the statistical relations of the quantum theory cannot be recovered in a 'cheap' way simply by introducing additional hypothetical variables to 'complete' the quantum theory. However, it does not seem to have been noticed that even Bohm's original 1951 theory was not a 'hidden variable theory' of this kind! The theory does not simply provide a 'realistic' description of a new kind of physical system with peculiar properties—the 'quantum object'. This interpretation of the theory is incompatible with the assignment of phase space probability distributions 'relative to the measurement context'. Part of the confusion is probably due to the term 'measurement', which is misleading here.†

Bohr often expressed dissatisfaction with his own writings on complementarity. In his article 'Discussion with Einstein on Epistemological Problems in Atomic Physics' Bohr referred to a passage explicating the notion of complementarity in his well-known reply to the Einstein–Podolski–Rosen paradox, as follows (Bohr, 1949b):

> 'Rereading these passages, I am deeply aware of the inefficiency of expression which must have made it very difficult to appreciate the trend of the argumentation *aiming to bring out the essential ambiguity involved in a reference to physical attributes of objects when dealing with phenomena where no sharp distinction can be made between the behaviour of the objects themselves and their interaction with the measuring instruments.* I hope, however, that the present account of the discussions with Einstein in the foregoing years, which contributed so greatly to make us familiar with the situation in quantum physics, may give a clearer impression of the *necessity of a radical revision of basic principles for physical explanation in order to restore logical order in this field of experience.*' (Italics inserted.)

The principle of complementarity, or the theorem of Kochen and Specker, is only the statement of the problem which the 'hidden variable' theories propose to resolve by 'a radical revision of basic principles for physical explanation', i.e., by suitably interpreting the physical parameters and statistical states of the quantum theory from the point of view of a broader theoretical and philosophical framework. The further elaboration of this broader description will be the subject of later articles.‡

References

Bell, J. S. (1966). *Reviews of Modern Physics*, **38**, 447.
Bohm, D. (1952a). *Physical Review*, **85**, 166, 180.
Bohm, D. (1952b). *Physical Review*, **85**, 187, 188.

† The irrelevance of a real '*measurement* problem' in the quantum theory has been discussed at length by Bohm in his unpublished paper presented to the Symposium: 'Quantum Theory and Beyond,' held at the University of Cambridge, England, July 15–20, 1968.

‡ See Bohm, D. (1965). 'Space, Time and Quantum Theory—Understood in terms of Discrete Structural Process', *Proceedings of the International Conference on Elementary Particles*. Kyoto.

Bohm, D. and Bub, J. (1966). *Reviews of Modern Physics*, **38**, 453. This article lists some references.

Bohr, N. (1949a). In: *Albert Einstein—Philosopher-Scientist*, ed. Schilpp, P. A. The Library of Living Philosophers, Inc., New York, p. 210

Bohr, N. (1949b). In: *Albert Einstein—Philosopher-Scientist*, p. 234.

Jauch, J. M. and Piron, C. (1963). *Helvetica physica acta*, **36**, 827.

Kochen, S. and Specker, E. P. (1967a). *Journal of Mathematics and Mechanics*, **17**, 59.

Kochen, S. and Specker, E. P. (1967b). *Journal of Mathematics and Mechanics*, **17**, 63.

Kochen, S. and Specker, E. P. (1967c). *Journal of Mathematics and Mechanics*, **17**, 81, 82.

Margenau, H. and Cohen, L. (1967a). In: *Quantum Theory and Reality*, ed. Bunge, M. Springer-Verlag, Inc., New York.

Margenau, H. and Cohen, L. (1967b). In: *Quantum Theory and Reality*, p. 88, equations (c_1), (c_2).

Margenau, H. and Cohen, L. (1967c). In: *Quantum Theory and Reality*, p. 88.

Moyal, J. E. (1949a). *Proceedings of the Cambridge Philosophical Society. Mathematical and Physical Sciences*, **45**, 99.

Moyal, J. E. (1949b). *Proceedings of the Cambridge Philosophical Society. Mathematical and Physical Sciences*, **45**, 101, 102.

Moyal, J. E. (1949c). *Proceedings of the Cambridge Philosophical Society. Mathematical and Physical Sciences*, **45**, 117, 123.

von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*, Chapter IV, Parts 1 and 2, pp. 295–328. Princeton University Press, Princeton.